HOSSAM GHANEM

(30) 8.8 *Improper Integrals (B)*

Example 1

Determine if the improper integral $\displaystyle\int_1^3 \frac{dx}{x^2\sqrt{x^2-1}}$ is convergent or divergent. Find its value if it is convergent.

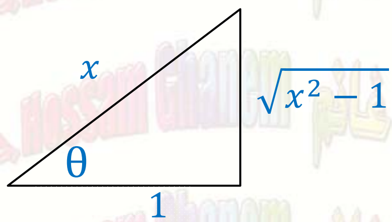
Solution

$$I_1 = \int \frac{1}{x^2\sqrt{x^2-1}}\,dx$$

$$x = \sec\theta \qquad\qquad dx = \sec\theta\tan\theta \; d\theta$$

$$\sec\theta = \frac{x}{1}$$

$$\theta = \sec^{-1} x$$



$$I_1 = \int \frac{1}{\sec^2\theta\,\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\; d\theta = \int \frac{1}{\sec^2\theta\tan\theta}\sec\theta\tan\theta\; d\theta$$

$$= \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + c = \frac{\sqrt{x^2-1}}{x} + c$$

$$I = \int_1^3 \frac{dx}{x^2\sqrt{x^2-1}} = \lim_{t\to1^+}\int_t^3 \frac{dx}{x^2\sqrt{x^2-1}} = \lim_{t\to1^+}\left[\frac{\sqrt{x^2-1}}{x}\right]_t^3 = \lim_{t\to1^+}\left[\frac{\sqrt{9-1}}{3} - \frac{\sqrt{t^2-1}}{t}\right] = \frac{\sqrt{8}}{3} - 0 = \frac{\sqrt{8}}{3}$$

I convergent

Example 2 *

37 August 7, 2010

Determine whether the improper integral $\displaystyle\int_0^\pi \frac{1+2\cos x}{x+2\sin x} - \frac{1}{x}\;dx$ is convergent or Divergent, and find its value if it is convergent (3 points)

Solution

$$I_1 = \int \frac{1+2\cos x}{x+2\sin x} - \frac{1}{x}\;dx = \ln|x+2\sin x| + \ln|x| + c = \ln\left|\frac{x+2\sin x}{x}\right| + c$$

$$I = \int_0^\pi \frac{1+2\cos x}{x+2\sin x} - \frac{1}{x}\;dx = \lim_{t\to0^+}\int_t^\pi \left(\frac{1+2\cos x}{x+2\sin x} - \frac{1}{x}\right)dx = \lim_{t\to0^+}\left[\ln\left|\frac{x+2\sin x}{x}\right|\right]_t^\pi$$

$$= \lim_{t\to0^+}\left[\ln\left|\frac{\pi+2\sin\pi}{\pi}\right| - \ln\left|\frac{t+2\sin t}{t}\right|\right] = \lim_{t\to0^+}\left[\ln\left|\frac{\pi+2\sin\pi}{\pi}\right| - \ln\left|\frac{t+2\sin t}{t}\right|\right]$$

$$= \ln\left|\frac{\pi+2\sin\pi}{\pi}\right| - \lim_{t\to0^+}\left[\ln\left|\frac{t+2\sin t}{t}\right|\right] = \ln\left(\frac{\pi}{\pi}\right) - \lim_{t\to0^+}\left[\ln\left|1+2\cdot\frac{\sin t}{t}\right|\right] = 0 - \ln 3 = -\ln 3$$

I convergent

Example 3*

36 June 6, 2010

(4 pts.) Determine whether the following improper integral is convergent or divergent. If it converges, find its value

$$\int_0^4 \frac{dx}{x^2+x-6}$$

Solution

$$I_1 = \int \frac{dx}{x^2+x-6}$$

$$x^2+x-6 = (x-2)(x+3)$$

$$\frac{1}{x^2+x-6} = \frac{1}{(x-2)(x+3)} = \frac{A}{(x-2)} + \frac{B}{(x+3)}$$

$$A(x+3) + B(x-2) = 1$$

at $x = 2$ $\quad 5A = 1 \quad A = \frac{1}{5}$

at $x = -3$ $\quad -5B = 1 \quad B = -\frac{1}{5}$

$$I_1 = \int \frac{dx}{x^2+x-6} = \int \left(\frac{\frac{1}{5}}{(x-2)} + \frac{-\frac{1}{5}}{(x+3)} \right) dx = \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3| + c = \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + c$$

$$I = \int_0^4 \frac{dx}{x^2+x-6} = \int_0^2 \frac{dx}{x^2+x-6} + \int_2^4 \frac{dx}{x^2+x-6}$$

$$I_2 = \int_0^2 \frac{dx}{x^2+x-6} = \lim_{t\to 2^-} \int_0^t \frac{dx}{x^2+x-6} = \lim_{t\to 2^-} \left[\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|\right]_0^t = \frac{1}{5}\lim_{t\to 2^-} \left[\ln\left|\frac{t-2}{t+3}\right| - \ln\left(\frac{2}{3}\right)\right] - \infty$$

I_2 divergent $\rightarrow I$ divergent

Example 4 Determine if the improper integral $\int_{-2}^{2} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx$ is convergent or divergent. Find its value if it is convergent.

Solution

$$I_1 = \int \frac{x^2}{(1-x)^{\frac{2}{3}}}dx$$

Let $u = 1-x \quad \rightarrow \quad x = 1-u \quad \rightarrow \quad dx = du$

$$I_1 = \int \frac{(1-u)^2}{u^{\frac{2}{3}}}dx = \int \frac{u^2-2u+1}{u^{\frac{2}{3}}}dx = \int \left(u^{\frac{4}{3}} - 2u^{\frac{1}{3}} + u^{-\frac{2}{3}}\right)dx = \frac{3}{7}u^{\frac{7}{3}} - \frac{3}{2}u^{\frac{4}{3}} + 3u^{\frac{1}{3}} + c$$

$$= \frac{3}{7}(1-x)^{\frac{7}{3}} - \frac{3}{2}(1-x)^{\frac{4}{3}} + 3(1-x)^{\frac{1}{3}} + c$$

$$I = \int_{-2}^{2} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx = \int_{-2}^{1} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx + \int_{1}^{2} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx$$

$$I_1 = \int_{-2}^{1} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx = \lim_{t\to1^-} \int_{-2}^{t} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx = \lim_{t\to1^-} \left[\frac{3}{7}(1-x)^{\frac{7}{3}} - \frac{3}{2}(1-x)^{\frac{4}{3}} + 3(1-x)^{\frac{1}{3}}\right]_{-2}^{t}$$

$$= \lim_{t\to1^-} \left[\left(\frac{3}{7}(1-t)^{\frac{7}{3}} - \frac{3}{2}(1-t)^{\frac{4}{3}} + 3(1-t)^{\frac{1}{3}}\right) - \left(\frac{3}{7}\cdot 3^{\frac{7}{3}} - \frac{3}{2}\cdot 3^{\frac{4}{3}} + 3\cdot 3^{\frac{1}{3}}\right)\right] = -\left(\frac{3}{7}\cdot 3^{\frac{7}{3}} - \frac{3}{2}\cdot 3^{\frac{4}{3}} + 3\cdot 3^{\frac{1}{3}}\right)$$

I_1 convergent

$$I_2 = \int_{1}^{2} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx = \lim_{t\to1^+} \int_{t}^{2} \frac{x^2}{(1-x)^{\frac{2}{3}}}dx = \lim_{t\to1^+} \left[\frac{3}{7}(1-x)^{\frac{7}{3}} - \frac{3}{2}(1-x)^{\frac{4}{3}} + 3(1-x)^{\frac{1}{3}}\right]_{t}^{2}$$

$$= \lim_{t\to1^-} \left[\left(\frac{3}{7}(-1)^{\frac{7}{3}} - \frac{3}{2}(-1)^{\frac{4}{3}} + 3(-1)^{\frac{1}{3}}\right) - \left(\frac{3}{7}(1-t)^{\frac{7}{3}} - \frac{3}{2}(1-t)^{\frac{4}{3}} + 3(1-t)^{\frac{1}{3}}\right)\right] = -\left(-\frac{3}{7} + \frac{3}{2} - 3^{\frac{1}{3}}\right)$$

I_2 convergent

I convergent

Example 5 Determine if the improper integral $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}dx$ is convergent or divergent. Find its value if it is convergent.

Solution

$$I_1 = \int \frac{1}{e^x + e^{-x}}dx = \int \frac{e^x}{e^x(e^x + e^{-x})}dx = \int \frac{e^x}{e^{2x}+1}dx$$

Let $u = e^x \qquad du = e^x dx$

$$I_1 = \int \frac{1}{u^2+1}du = \tan^{-1}u + c = \tan^{-1}e^x + c$$

$$I = \int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}dx = \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}}dx + \int_{0}^{\infty} \frac{1}{e^x + e^{-x}}dx$$

$$I_2 = \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}}dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{e^x + e^{-x}}dx = \lim_{t\to-\infty} \left[\tan^{-1}e^x\right]_{t}^{0} = \lim_{t\to-\infty} [\tan^{-1}1 - \tan^{-1}e^t] = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$I_3 = \int_{0}^{\infty} \frac{1}{e^x + e^{-x}}dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{e^x + e^{-x}}dx = \lim_{t\to\infty} \left[\tan^{-1}e^x\right]_{0}^{t} = \lim_{t\to\infty} [\tan^{-1}e^t - \tan^{-1}1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$I = I_2 + I_3 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

I convergent

Example 6
56 11 December 2011

Determine if the improper integral $\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx$ ($2\,^1/_2$ points)

is convergent or divergent. Find its value if it is convergent.

Solution

$$I_1 = \int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$I = \int_{-\infty}^{\infty} \frac{1}{4+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{4+x^2}\,dx + \int_{0}^{\infty} \frac{1}{4+x^2}\,dx$$

$$I_2 = \int_{-\infty}^{0} \frac{1}{4+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{4+x^2}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_t^0 = \frac{1}{2}\lim_{t\to-\infty}\left[\tan^{-1}0 - \tan^{-1}\frac{t}{2}\right] = \frac{1}{2}\left(0+\frac{\pi}{2}\right) = \frac{\pi}{4}$$

$$I_3 = \int_{0}^{\infty} \frac{1}{4+x^2}\,dxdx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{4+x^2}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{2} - \tan^{-1}0\right] = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}$$

$$I = I_2 + I_3 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \qquad I \text{convergent}$$

Homework

Determine whether the following improper integrals convergent or divergent , and find its value if it convergent

1	$\int_{-1}^{1} \frac{1}{\cosh x - 1} dx$	9	$\int_{-3}^{0} \frac{x}{(1-x^2)^2} dx$	17	$\int_{1}^{\infty} \frac{1}{x + x^3} dx$
2	$\int_{0}^{\infty} \frac{dx}{\sqrt{1+\sinh^2 x}}$	10	$\int_{0}^{\infty} \frac{x^2}{(1+x^3)^2} dx$	18	$\int_{0}^{1} \frac{1}{\sqrt[3]{1-x}} dx$
3	$\int_{1}^{\infty} \frac{\tan^{-1} x}{x^2} dx$	11	$\int_{0}^{4} (2-x)^{-\frac{3}{4}} dx$	19	$\int_{1}^{3} \frac{dx}{x^2\sqrt{x^2-1}}$
4	$\int_{3}^{\infty} \frac{1}{x^2 - 2x - 3} dx$	12	$\int_{0}^{1} \frac{1}{\sqrt[3]{1-x}} dx$	20	$\int_{1}^{\infty} \frac{1}{x + x^3} dx$
5	$\int_{2}^{\infty} \frac{1}{x^2 - 2x + 4} dx$	13	$\int_{-\infty}^{3} \frac{dx}{\sqrt{4-x}}$	21	
6	$\int_{1}^{\infty} \frac{1}{x^2 - 2x + 5} dx$	14	$\int_{1}^{3} \frac{dx}{x^2\sqrt{x^2-1}}$	22	
7	$\int_{1}^{2} \frac{x}{3x^2 - 6x + 4} dx$	15	$\int_{1}^{\infty} \frac{dx}{x(x^2+1)}$	23	
8	$\int_{-\infty}^{\infty} \frac{1}{4x^2 + 4x + 5} dx$	16	$\int_{0}^{\infty} \frac{x}{(x+1)(x^2+1)} dx$	24	

Homework

1	51 May 13, 2010 Determine whether the following improper integral is convergent or divergent. if it convergent, find its value. [2 pts. each] (a) $\int_0^1 \frac{dx}{(x+1)\sqrt{x}}$ (b) $\int_1^\infty \frac{dx}{(x+1)\sqrt{x}}$
2	54 12/05/2011 Show that the improper integral $\int_1^2 \frac{1}{x\sqrt{4-x^2}}\,dx$ converges to $\ln\sqrt{2+\sqrt{3}}$
3	35 January 24, 2010 Determine whether the improper integral $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$ is convergent or divergent if converges, find its value. (4 pts.)
4	38 Jan. 22, 2011 (4 pts.) Determine whether the following improper integral is convergent or divergent. if it converges, find its value. $\int_{-\infty}^{-2} \frac{x+1}{x^2+2x}\,dx$
5	Evaluate the following $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$ ($2\,{}^1/_2$ points) 56 11 December 2011
6	Evaluate $\int_0^\pi \frac{\sec^2 x}{4+\tan^2 x}\,dx$ (4 pts) 41 14 January 2012